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No Phase Transition for Gaussian Fields with Bounded Spins

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Abstract Let $a < b$, $\Omega = [a,b]^{\mathbb{Z}^d}$ and H be the (formal) Hamiltonian defined on Ω by

$$H(\eta) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} J(x-y)(\eta(x)-\eta(y))^2, \tag{1}$$

where $J : \mathbb{Z}^d \to \mathbb{R}$ is any summable non-negative symmetric function ($J(x) \geq 0$ for all $x \in \mathbb{Z}^d$, $\sum_x J(x) < \infty$ and $J(x) = J(-x)$). We prove that there is a unique Gibbs measure on Ω associated to H. The result is a consequence of the fact that the corresponding Gibbs sampler is attractive and has a unique invariant measure.

Keywords Truncated Gaussian fields · Bounded spins · Quadratic potential · No phase transition

1 Introduction

Let $\Omega = [a,b]^{\mathbb{Z}^d}$. Let the function $J : \mathbb{Z}^d \to \mathbb{R}^+$ be summable, non-negative and symmetric: $J(x) \geq 0$ for all $x \in \mathbb{Z}^d$ and $0 < \|J\| := \sum_x J(x) < \infty$; it is convenient to also assume $J(0) = 0$. For each finite $\Lambda \subset \mathbb{Z}^d$, consider the "ferromagnetic" Hamiltonian $H^\Lambda : \Omega \to \mathbb{R}$ given by the quadratic potential

$$H^\Lambda(\eta) := \frac{1}{2} \sum_{\{x,y\}\not\subset\Lambda^c} J(y-x)(\eta(x)-\eta(y))^2. \tag{2}$$

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Let the *specification* $\Gamma = \{\mu^{\Lambda,\gamma} : \Lambda \subset \mathbb{Z}^d \text{ finite}, \gamma \in \Omega\}$ be the family of local measures induced by H^Λ: for finite Λ and $\gamma \in \Omega$ let $\mu^{\Lambda,\gamma}$ be the measure on $[a,b]^\Lambda$ with boundary conditions γ defined by

$$\mu^{\Lambda,\gamma}(d\eta_\Lambda) := \frac{1}{Z^{\Lambda,\gamma}} \exp(-H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c}))d\eta_\Lambda, \tag{3}$$

where $Z^{\Lambda,\gamma}$ is the normalizing constant and $(\eta_\Lambda\gamma_{\Lambda^c}) \in \Omega$ is the *juxtaposition* of η_Λ and γ_{Λ^c}:

$$\eta_\Lambda\gamma_{\Lambda^c}(x) = \begin{cases} \eta(x) & \text{if } x \in \Lambda, \\ \gamma(x) & \text{if } x \in \Lambda^c. \end{cases}$$

The measures $\mu^{\Lambda,\gamma}$ are called *kernels*.

A *Gibbs measure compatible* with Γ is a measure μ on Ω satisfying the "DLR" (Dobrushin, Lanford and Ruelle [4, 5]) equations

$$\int \mu(d\gamma) \int \mu^{\Lambda,\gamma}(d\eta_\Lambda) f(\eta_\Lambda\gamma_{\Lambda^c}) = \int \mu(d\eta) f(\eta), \tag{4}$$

for continuous $f : \Omega \to \mathbb{R}$. Using the notation $\mu f = \int \mu(d\eta) f(\eta)$, the DLR equations read

$$\mu(\mu^{\Lambda,(\cdot)} f) = \mu f. \tag{5}$$

We prove that for this model there exists a unique Gibbs measure:

Theorem 1 *Let $J : \mathbb{Z}^d \to \mathbb{R}^+$ be summable non-negative symmetric function such that $0 < \|J\| < \infty$. Let $\Gamma = \{\mu^{\Lambda,\gamma} : \Lambda \in \mathcal{S}, \gamma \in \Omega\}$ be the specification* (3) *induced by the Hamiltonian* (2). *Then there exists a unique Gibbs measure compatible with Γ.*

This theorem is proven at the end of Sect. 3. The result is not new. It has been proven by Mac Bryan and Spencer [2], see the review paper of Bricmont, Mellouki and Froehlich [1]. Those references prove that under any Gibbs measure, the correlations between the heights at sites 0 and x decrease exponentially with x; the uniqueness of the Gibbs state is then a consequence of an argument by Sokal quoted in [1]. Section 4 of the lecture notes of Velenik [14] reports the asymptotics of the "mass" and the variance of the height at a given site in function of the size $b-a$. Recently Sakagawa [12] shows that in the nearest neighbor case ($J(x-y) = 0$ for $|x-y| > 1$) in dimension $d \geq 3$ the unique Gibbs state remains localized at the center of the box $(a+b)/2$ even when $b-a$ goes to infinity. Lee [10] establishes the asymptotics of the maximal absolute value of the heights when Λ_N is a cube of side $2N$, as $N \to \infty$.

The references quoted above treat directly the properties of the specification Γ; the proofs are relatively involved. Our approach uses the fact that Gibbs measures are invariant for the Gibbs sampler, a Markov process used to simulate the Gibbs measures via Markov Chain Monte Carlo methods. For positive J the Gibbs sampler is attractive, which roughly speaking means that ordered configurations remain ordered as time grows. Attractiveness implies in this case that there is a unique invariant measure for the Gibbs sampler in infinite volume.

The Gibbs measure constructed in the infinite interval $(a,b) = (-\infty, \infty)$ are referred to as *free field*. In one and two dimensions there are no (infinite volume) Gibbs measures; this is called *delocalization*. In dimensions three and bigger there is a unique ergodic Gibbs measure μ such that $\int \mu(d\eta)\eta(x) = 0$ for all $x \in \mathbb{Z}^d$; this is *localization*. For each function

h harmonic for J ($\sum_y J(x,y)h(y) = h(x)\sum_y J(x,y)$) the law of $\eta + h$ is also a Gibbs measure when η is distributed according to μ. In particular, h may be a "hyperplane" $h(x) = \langle \alpha, x\rangle + c$ for the vector $\alpha \in \mathbb{R}^d$ and $c \in \mathbb{R}$, where $\langle\cdot,\cdot\rangle$ is internal product. The absence of phase transition when the spins are bounded may be a consequence of the "entropic repulsion": the entropy of surfaces located around $(a+b)/2$ is higher than the surfaces located closer to one of the extremes of the interval; particularly striking in this direction is the result of Sakagawa [12] described above. Localization and delocalization for free fields are discussed in Sect. 1 of Velenik [14] from a static point of view and by Hsiao [7, 8] and Ferrari and Niederhauser [3] with a dynamical approach in the spirit of this paper.

Since $H^\Lambda(\eta) = \|J\| H^\Lambda(\eta/\sqrt{\|J\|})$, where $(\eta/c)(x) = \eta(x)/c$ for all x and the interval $[a,b]$ is arbitrary, we can and will assume

$$\|J\| = \sum_{x\in\mathbb{Z}^d} J(x) = 1 \tag{6}$$

without losing generality. In fact, if we choose $\|J\| = 1$ and introduce an inverse temperature β defining

$$\mu_\beta^{\Lambda,\gamma}(d\eta_\Lambda) := \frac{1}{Z_\beta^{\Lambda,\gamma}} \exp(-\beta H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c}))d\eta_\Lambda, \tag{7}$$

we have $\beta H^\Lambda(\eta) = H^\Lambda(\sqrt{\beta}\eta)$. If $\eta \in [a^*, b^*]^{\mathbb{Z}^d}$, then $\sqrt{\beta}\eta \in [\sqrt{\beta}a^*, \sqrt{\beta}b^*]^{\mathbb{Z}^d}$. Since Theorem 1 is true for any interval, substituting $[a,b]$ with $[\sqrt{\beta}a, \sqrt{\beta}b]$ we obtain that the model at inverse temperature β and spins in $[a,b]$ has a unique Gibbs measure. It is then sufficient to consider the case $\beta = 1$ because the other cases reduce to this one.

Antiferromagnetic Case in Bipartite Graphs The usual trick permits to extend Theorem 1 to negative J in bipartite graphs. Assume J satisfies the conditions of Theorem 1 and $J(x-y) = 0$ if $x, y \in \Upsilon_1$ or $x, y \in \Upsilon_2$ for a partition Υ_1, Υ_2 of $\mathbb{Z}^d$. Define $\tilde{J}(x) = -J(x)$ and $\tilde{\Gamma}$ the specification constructed with $\tilde{J}$. Define the transformation $(R\eta)(x) = \eta(x)$ for $x \in \Upsilon_1$ and $(R\eta)(x) = a + b - \eta(x)$ for $x \in \Upsilon_2$. For a measure μ on Ω, call $R\mu$ the measure induced by this transformation. Then μ is Gibbs for Γ if and only if $R\mu$ is Gibbs for $\tilde{\Gamma}$. This implies that Theorem 1 holds also for the specification $\tilde{\Gamma}$.

2 Stochastic Domination and Gibbs Sampler

In this section we collect some general known results about stochastic domination, introduce the Gibbs sampler process and discuss properties of the set of invariant measures for the Gibbs sampler related to attractiveness of the process. The particular form of the specification is not relevant here. Most results are easy extensions to the continuous space Ω of results of Chaps. 3 and 4 of Liggett [11] for the space $\{0,1\}^{\mathbb{Z}^d}$.

Stochastic Domination in Ω For $\eta, \xi \in \Omega$ say that $\eta \le \xi$ if and only if $\eta(x) \le \xi(x)$ for all $x \in \mathbb{Z}^d$. A function $f : \Omega \to \mathbb{R}$ is increasing if and only if $f(\eta) \le f(\xi)$ for $\eta \le \xi$. Let μ_1 and μ_2 probability measures on Ω. We say that μ_2 dominates stochastically μ_1, and denote $\mu_1 \preceq \mu_2$, if $\mu_1 f \le \mu_2 f$ for each increasing measurable function f. $\mu_1 \preceq \mu_2$ if there exists a coupling $(\hat{\eta}_1, \hat{\eta}_2)$ with marginals μ_1 and μ_2 such that $\hat{\eta}_1 \le \hat{\eta}_2$ almost surely [11, 13].

Gibbs Sampler The Gibbs sampler associated to a specification Γ is a continuous time Markov process $(\eta_t : t \geq 0)$ on Ω with infinitesimal generator L defined on cylinder continuous functions $f : \Omega \to \mathbb{R}$ by:

$$Lf(\eta) := \sum_{x\in\mathbb{Z}^d} L_x f(\eta), \quad L_x f(\eta) := \int_a^b \mu^{\{x\},\eta}(ds)[f(\eta + (s - \eta(x))\theta_x) - f(\eta)], \tag{8}$$

where $\theta_x \in \{0,1\}^{\mathbb{Z}^d}$ is defined by $\theta_x(x) = 1$ and $\theta_x(z) = 0$ for $z \neq x$. In words, at rate 1, at each site $x \in \mathbb{Z}^d$ the spin $\eta(x) \in [a,b]$ is updated with the law $\mu^{\{x\},\eta}$. The existence of a process η_t with generator L such that $\frac{d}{dt}\mathbb{E}(f(\eta_t)|\eta_0 = \eta) = Lf(\eta)$ is standard, using Harris [6] graphical construction and a percolation argument. Call $S(t)$ the corresponding semigroup defined by $S(t)f(\eta) = \mathbb{E}(f(\eta_t)|\eta_0 = \eta)$. The semigroup acts on measures via the formula $(\mu S(t))f = \mu(S(t)f)$; $\mu S(t)$ is the law of the process at time t when the initial distribution is μ. We say that μ is invariant for the process if $\mu S(t) = \mu$. A measure μ is invariant if and only if $\mu Lf = 0$ for all continuous cylinder f.

Proposition 2 *If a measure μ is Gibbs for specification Γ then it is invariant for the Gibbs sampler associated to Γ.*

Proof It suffices to show $\mu L_x f = 0$ for all $x \in \mathbb{Z}^d$ and continuous cylinder f.

$$\begin{aligned} \mu L_x f &= \int \mu(d\eta) \int_a^b \mu^{\{x\},\eta}(ds)[f(\eta + (s - \eta(x))\theta_x) - f(\eta)] \\ &= \mu(\mu^{\{x\},(\cdot)} f) - \mu f = 0, \end{aligned} \tag{9}$$

by (5). □

Attractiveness A process is *attractive* if $\mu_1 \preceq \mu_2$ implies $\mu_1 S(t) \preceq \mu_2 S(t)$. A sufficient condition for attractiveness of the Gibbs sampler is

$$\mu^{\{x\},\eta} \preceq \mu^{\{x\},\xi} \quad \text{if } \eta \leq \xi. \tag{10}$$

Let δ^a and δ^b be the measures concentrating mass on the configuration “all a” and “all b” respectively. Clearly, $\delta^a \preceq \mu \preceq \delta^b$ for any measure μ. If the process is attractive, $\delta^b S(t)$ is non increasing and $\delta^a S(t)$ is non decreasing in t. Hence both sequences have a (weak) limit when $t \to \infty$ that we call μ^b and μ^a, respectively. Both μ^b and μ^a are invariant measures called upper and lower invariant measures respectively. For any measure μ attractiveness implies $\delta^a S(t) \preceq \mu S(t) \preceq \delta^b S(t)$ for all t. If μ is invariant $\mu S(t) = \mu$ for all t and taking limits as $t \to \infty$,

$$\mu^a \preceq \mu \preceq \mu^b. \tag{11}$$

Proposition 3 *Assume the Gibbs sampler associated to Γ is attractive and $\mu^a = \mu^b$. Then if μ is a Gibbs measure compatible with Γ, $\mu = \mu^a = \mu^b$.*

Proof By Proposition 2 Gibbs measures are invariant for the Gibbs sampler, hence any Gibbs measure μ must satisfy (11), showing uniqueness. □

3 Truncated Gaussian Fields and Gibbs Sampler

In this section we discuss some basic properties of truncated Gaussian variables, show that the truncated Gaussian Gibbs sampler is attractive and that the upper and lower invariant measures for the Gibbs sampler coincide, proving Theorem 1.

Truncated Normal Variables For $a < b$ and $m \in \mathbb{R}$, let the *truncated normal distribution* $\mathcal{N}_{a,b}(m,1)$ be the distribution on $[a,b]$ with density g_m given by

$$g_m(u) := \frac{\phi(u-m)}{\Phi(b-m)-\Phi(a-m)}, \quad a \le u \le b, \tag{12}$$

where $\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$ is the standard normal distribution and $\Phi(u) = \int_{-\infty}^{u} \phi(s)ds$ is the cumulative distribution. The truncated normal $\mathcal{N}_{a,b}(m,1)$ is just the normal $\mathcal{N}(m,1)$ conditioned to the interval $[a,b]$. The cumulative distribution function is given by $G_m(u) := \int_a^u g_m(y)dy$.

Lemma 4 *If $a \le m_1 < m_2 \le b$, then $G_{m_1}(u) \ge G_{m_2}(u)$, for $u \in [a,b]$. In particular $\mathcal{N}_{a,b}(m_1,1) \preceq \mathcal{N}_{a,b}(m_2,1)$.*

Proof Writing

$$\log g_{m_2}(u) - \log g_{m_1}(u) = u(m_2-m_1) - \frac{m_2^2-m_1^2}{2} + \log\left(\frac{\Phi(b-m_1)-\Phi(a-m_1)}{\Phi(b-m_2)-\Phi(a-m_2)}\right), \tag{13}$$

we see that there is a unique solution of $g_{m_2}(v) - g_{m_1}(v) = 0$, which is called $v(m_1,m_2)$. Then,

$$g_{m_2}(u) \le g_{m_1}(u) \quad \text{if and only if} \quad u \le v(m_1,m_2). \tag{14}$$

Since $g_{m_2}(u) - g_{m_1}(u)$ is the derivative with respect to u of $G_{m_2}(u) - G_{m_1}(u)$ and $G_{m_2}(a) - G_{m_1}(a) = G_{m_2}(b) - G_{m_1}(b) = 0$, (14) implies $G_{m_2}(u) \le G_{m_1}(u)$ for $u \in [a,b]$. □

Johnson and Kotz (Sect. 7 of Chap. 13 in [9]) prove that the mean of $\mathcal{N}_{a,b}(m,1)$ is given by

$$\int_a^b s g_m(s)ds = m - \varphi(m), \tag{15}$$

where

$$\varphi(m) := \frac{\phi(b-m)-\phi(a-m)}{\Phi(b-m)-\Phi(a-m)}. \tag{16}$$

The function φ is odd with respect to $\frac{a+b}{2}$: $\varphi\left(\frac{a+b}{2}+m\right) = -\varphi\left(\frac{a+b}{2}-m\right)$ for all $0 \le m \le \frac{b-a}{2}$. Furthermore φ is increasing, continuous and invertible in the interval $a \le m \le b$.

Truncated Gaussian Gibbs Sampler The kernel $\mu^{\{x\},\eta}$ given by (3) with $\|J\| = 1$ is a truncated normal distribution $\mathcal{N}_{a,b}(\overline{\eta}(x),1)$, where

$$\overline{\eta}(x) = \sum_{y \ne x} J(y-x)\eta(y).$$

Since $\eta \le \xi$ implies $\overline{\eta}(x) \le \bar{\xi}(x)$, Lemma 4 implies $\mathcal{N}_{a,b}(\overline{\eta}(x), 1) \preceq \mathcal{N}_{a,b}(\bar{\xi}(x), 1)$, that is, $\mu^{\{x\},\eta} \preceq \mu^{\{x\},\xi}$. But this is condition (10) which implies that the corresponding Gibbs sampler is attractive. Furthermore for $x \in \mathbb{Z}^d$ and $f(\eta) = \eta(x)$,

$$Lf(\eta) = \int_a^b \mu^{\{x\},\eta}(ds)(s - \eta(x)) = \overline{\eta}(x) - \varphi(\overline{\eta}(x)) - \eta(x), \tag{17}$$

using (15). We abuse notation writing $\overline{\eta}(x)$ and $\eta(x)$ instead of h and f, for $h(\eta) = \overline{\eta}(x)$ and $f(\eta) = \eta(x)$.

Lemma 5 *Let μ be invariant for the Gibbs Sampler and translation invariant. Then*

$$\mu\varphi(\overline{\eta}(x)) = 0. \tag{18}$$

Proof Since μ is invariant for the Gibbs Sampler, taking expectation with respect to μ in (17),

$$0 = \mu L(\eta(x)) = \mu(\overline{\eta}(x)) - \mu(\varphi(\overline{\eta}(x))) - \mu(\eta(x)). \tag{19}$$

On the other hand, by translation invariance, $\mu(\eta(x))$ does not depend on x. Hence,

$$\mu(\overline{\eta}(x)) = \mu(\eta(x)) \sum_{y:y\neq x} J(y-x) = \mu(\eta(x)) \tag{20}$$

(recall $\sum_{y\neq 0} J(y) = 1$). □

Proof of Theorem 1 Existence of a Gibbs measure μ is proven in Chap. 4 of [4] as $[a,b]$ is a *standard Borel space* of finite measure and the potential J is absolutely summable.

Since the Gibbs sampler is attractive, the upper and lower invariant measures μ^b and μ^a are well defined. By Proposition 3 it suffices to show $\mu^a = \mu^b$. Let (η^a, η^b) be a random vector with marginals μ^a and μ^b and such that $\eta^a \le \eta^b$. The function $\overline{\eta}(x)$ is increasing in η and $\varphi(m)$ is increasing in m. Hence

$$\varphi(\overline{\eta}^a(x)) \le \varphi(\overline{\eta}^b(x)). \tag{21}$$

Since the limit defining μ^a and μ^b is translation invariant, so are μ^a and μ^b and by (18), $\varphi(\overline{\eta}^a(x))$ and $\varphi(\overline{\eta}^b(x))$ have expected value 0. This and (21) imply $\varphi(\overline{\eta}^a(x)) = \varphi(\overline{\eta}^b(x))$ a.s. Since φ is invertible, $\overline{\eta}^a(x) = \overline{\eta}^b(x)$ a.s. That is,

$$\sum_{y:y\neq x} J(y-x)(\eta^b(y) - \eta^a(y)) = 0. \tag{22}$$

Since $\eta^a \le \eta^b$, (22) implies $\eta^b(y) = \eta^a(y)$ for all y such that $J(y-x) > 0$. Since x is arbitrary, this implies $\eta^a(y) = \eta^b(y)$ almost surely for all y. □

4 Specification Kernels are Truncated Multivariate Normal Distributions

In this section (which can be read independently of the others, except for notation) we show that the specification kernels are truncated multivariate normal distributions.

Lemma 6 *For each finite* Λ *and* $\gamma \in \Omega$, *the kernel* $\mu^{\Lambda,\gamma}$ *is a multivariate Normal distribution* $\mathcal{N}_\Lambda(\mathbf{m}_\Lambda^\gamma, \mathbf{\Sigma}_\Lambda)$ *truncated to the box* $[a,b]^\Lambda$, *where*

$$\mathbf{m}_\Lambda^\gamma = (A^\Lambda)^{-1} B^{\Lambda,\Lambda^c} \gamma_{\Lambda^c} \quad \textit{and} \quad \mathbf{\Sigma}_\Lambda = (A^\Lambda)^{-1} \tag{23}$$

with

$$A^\Lambda(x,y) := \begin{cases} \sum_{y\in\Lambda\setminus\{x\}} J(y-x) + \|J\|, & \textit{if } x = y \in \Lambda, \\ -J(y-x), & \textit{if } x \in \Lambda \textit{ and } y \in \Lambda\setminus\{x\} \end{cases} \tag{24}$$

and

$$B^{\Lambda,\Lambda^c}(x,y) := J(y-x), \quad \textit{for } x\in\Lambda \textit{ and } y \in \Lambda^c. \tag{25}$$

Proof A simple computation shows that for H^Λ defined by (2),

$$H^\Lambda(\eta) = \frac{1}{2}(\eta'_\Lambda A^\Lambda \eta_\Lambda - 2\eta'_\Lambda B^{\Lambda,\Lambda^c}\eta_{\Lambda^c} + \Psi(\eta_{\Lambda^c})), \tag{26}$$

where the function $\Psi(\eta_{\Lambda^c}) = \sum_{x\in\Lambda}\sum_{y\in\Lambda^c} J(y-x)\eta(y)^2$ does not depend on η_Λ. If A^Λ is positive definite, this shows the proposition because, using (26),

$$\begin{aligned} H^\Lambda(\eta_\Lambda\gamma_{\Lambda^c}) &= \frac{1}{2}(\eta'_\Lambda A^\Lambda \eta_\lambda - 2\eta'_\Lambda A^\Lambda((A^\Lambda)^{-1}B^{\Lambda,\Lambda^c}\gamma_{\Lambda^c}) + \Psi(\gamma_{\Lambda^c})) \\ &= \frac{1}{2}(\eta_\Lambda - \mathbf{m}_\Lambda^\gamma)' A^\Lambda(\eta_\Lambda - \mathbf{m}_\Lambda^\gamma) + R(\gamma), \end{aligned}$$

where $\mathbf{m}_\Lambda^\gamma = (A^\Lambda)^{-1}B^{\Lambda,\Lambda^c}\gamma_{\Lambda^c}$ and $R(\gamma)$ does not depend on η_Λ.

If J satisfies the conditions of Theorem 1, then A^Λ is positive definite. Indeed, A^Λ can be decomposed as the sum of a positive semidefinite matrix and a linear combination of positive matrices, as follows

$$A^\Lambda = \sum_{x\in\Lambda}\Big(\sum_{y\in\Lambda\setminus\{x\}} J(y-x)\Big)E_\Lambda^{xx} + \Big(\sum_{z\in\mathbb{Z}_+^d\setminus\Delta_+} 2J(z)\Big)I_\Lambda + \sum_{z\in\Delta_+} J(z)T_z^\Lambda, \tag{27}$$

where I_Λ is the identity matrix and $T_z^\Lambda : z \in \{y-x : (x,y)\in\Lambda\times\Lambda\}\setminus\{0\}$ is given by

$$T_z^\Lambda(x,y) = 2\mathbf{1}\{x=y\} - \mathbf{1}\{y-x\in\{-z,z\}\}, \tag{28}$$

$E_\Lambda^{xx}(z,w) = \mathbf{1}\{(z,w) = (x,x)\}$ and $(\mathbb{Z}_+^d, \mathbb{Z}_-^d)$ is a partition of $\mathbb{Z}^d\setminus\{0\}$ such that $x\in\mathbb{Z}_+^d \Leftrightarrow -x\in\mathbb{Z}_-^d$ and $\Delta_+ = \Delta\cap\mathbb{Z}_+^d$.

Finally, let's prove that for each $z\in\Delta_+$, the matrix T_z^Λ given by (28) is positive definite. We say that sites $x,y\in\Lambda$ are *z-connected* if there exists $x = x_0, x_1, \ldots, x_n = y$ in Λ such that $x_m - x_{m-1} \in \{-z, z\}$ for all $m = 1,\ldots,n$. Since z-connected is an equivalence relation, Λ is decomposed in the equivalent classes $\Lambda_1,\ldots,\Lambda_n$ given by $\Lambda_\ell = \{x_\ell + mz : m = 0,\ldots,m_\ell\}$ for some m_ℓ non-negative integer.

Take a non-null vector $\eta\in\mathbb{R}^\Lambda$ and use the previous notation to get

$$\eta' T_z^\Lambda \eta = \sum_{\ell:m_\ell\geq 1}\sum_{m=0}^{m_\ell-1}(\eta(x_\ell+mz) - \eta(x_\ell+(m+1)z))^2$$

$$+2\sum_{\ell:m_\ell=0}\eta(x_\ell)^2+\sum_{\ell:m_\ell\geq 1}(\eta(x_\ell)^2+\eta(x_\ell+m_\ell z)^2)>0. \tag{29}$$

This proves that T_z^Λ is positive definite and the lemma. □

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References

1. Bricmont, J., El Mellouki, A., Fröhlich, J.: Random surfaces in statistical mechanics: roughening, rounding, wetting. J. Stat. Phys. **42**(5–6), 743–798 (1986)
2. McBryan, O.A., Spencer, T.: On the decay of correlations in SO(n)-symmetric ferromagnets. Commun. Math. Phys. **53**(3), 299–302 (1977)
3. Ferrari, P.A., Niederhauser, B.M.: Harness processes and harmonic crystals. Stoch. Process. Appl. **6**, 939–956 (2006). math.PR/0312402
4. Georgii, H.O.: Gibbs Measures and Phase Transitions. de Gruyer, Berlin (1988)
5. Georgii, H.O., Häggström, O., Maes, C.: The random geometry of equilibrium phases. In: Phase Transitions and Critical Phenomena, vol. 18, pp. 1–142. Academic Press, San Diego (2001)
6. Harris, T.E.: Additive set-valued Markov processes and graphical methods. Ann. Probab. **6**(3), 355–378 (1978)
7. Hsiao, C.-T.: Stochastic processes with Gaussian interaction of components. Z. Wahrscheinlichkeitstheor. Verw. Geb. **59**, 39–53 (1982)
8. Hsiao, C.-T.: Infinite systems with locally additive interaction of components. Chin. J. Math. **18**(2), 83–95 (1985)
9. Johnson, N.L., Kotz, S.: Continuous Univariate Distributions, vol. 1. Wiley, New York (1970)
10. Lee, D.-S.: Distribution of extremes in the fluctuations of two-dimensional equilibrium interfaces. Phys. Rev. Lett. **95**, 150–601 (2005)
11. Liggett, T.M.: Interacting Particle Systems. Springer, New York (1985)
12. Sakagawa, H.: Entropic repulsion for the high dimensional Gaussian lattice field between two walls. J. Stat. Phys. **124**(5), 1255–1274 (2006)
13. Thorisson, H.: Coupling, Stationarity, and Regeneration. Springer, New York (2000)
14. Velenik, Y.: Localization and delocalization of random interfaces. Probab. Surv. **3**, 112–169 (2006) (electronic)